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A General Dynamic Equilibrium Model and Business Cycles

Prof. Wei-Bin ZHANG
Ritsumeikan Asia Pacific University, Japan
wbz1@apu.ac.jp

ABSTRACT: This study generalizes the economic growth model of heterogeneous households proposed by Zhang (2017). Zhang's model is built on Ricardian theory of distribution, Walrasian general equilibrium theory, and neoclassical growth theory. The heterogeneous-household economy is composed of a consumer goods sector, an agricultural goods sector, and one capital goods sector. Technology, land, population and human capital are exogenous. This paper generalizes Zhang's model by allowing constant coefficients to be time-dependent. We show the existence of business cycles due to different exogenous oscillatory changes.

KEYWORDS: business cycle, periodic shocks, general equilibrium theory, neoclassical growth theory, integrated theory

JEL Classification: D31, O41

1. Introduction

Zhang (2017) recently proposes a growth model which is built on micro behavior to reveal complexity of macroeconomic phenomena. This study generalizes Zhang's economic growth model with economic structure and heterogeneous households. Different from most of the growth models with endogenous wealth, the model takes account of land distribution and determines endogenously land value. The model is built by integrating the main determinants of economic growth in the three important theories in economics - Walrasian general equilibrium theory, Ricardian theory of distribution, and neoclassical growth theory.

Walrasian general equilibrium theory of pure exchange and production economies has been well developed over more than one hundred years (Walras 1874; Arrow and Debreu 1954; Debreu 1959; Arrow and Hahn 1971; and Mas-Colell *et al.* 1995). The general equilibrium theory reveals economic mechanisms of production, consumption, and exchanges when durable variables, such as resources, wealth, and environment, are exogenously given. The theory fails to deal with issues related to, for instance, endogenous wealth accumulation (e.g., Impicciatore et al. 2012). This study models behavior of households with wealth accumulation. Ricardian theory of distribution shows how that wages, interest rate, and rent can be determined within a compact theory (Ricardo 1821). This study applies Ricardian theory in the sense that it includes land and endogenous determination of land rent in Walrasian theory (Samuelson 1959; Barkai 1959; Pasinetti 1960; Negish 1989; Morishima 1989). It is well known that neither the Walrasian theory nor Ricardian theory provides a profound microeconomic mechanism of wealth accumulation. Neoclassical growth theory models endogenous wealth accumulation on the basis of microeconomic mechanism (e.g., Solow 1956; Burmeister and Dobell 1970;

Jensen and Larsen 2005; Zhang 2005a). Zhang (2017) integrates the economic mechanisms of the three theories in economics into an integrated analytical framework. This study generalizes Zhang's model by allowing constant coefficients to be time-dependent. The generalization makes the original model more robust as it can analyze any impact of exogenous changes on the economic behavior. The paper also makes a contribution to the literature of business cycles (Zhang 1991, 2005, 2006; Lorenz 1993; Chiarella and Flaschel 2000; Shone 2002; Gandolfo 2005; Puu 2011; Stachurski et al. 2014). The rest of the paper is organized as follows. Section 2 builds generalizes Zhang's model by allowing constant coefficients to be time-dependent. Section 3 studies properties of the heterogeneous-household model and simulates the model for an economy of three types of households with all the coefficients being constant. Section 4 shows the existence of business cycles to different periodic exogenous shocks. Section 5 concludes the study.

2. The basic model

This section generalizes Zhang's model merely by allowing constant coefficients to be time-dependent. We refer explanations of modelling in detail to Zhang (2017). The economy is composed of agricultural, capital goods and consumer goods sector. We generalize the Uzawa two sector model by adding agricultural sector (Uzawa 1961). Neoclassical growth theory is the analytical framework upon which we describe markets and production sectors. Households own the assets of the economy. We apply Zhang's concept of disposable income and utility function to describe human behavior. Households use up their disposable incomes to consume and save. All the markets are perfectly competitive. All factors are fully utilized. The population is composed of multiple groups. Each group has population, $\bar{N}_j(t)$, ($j = 1, ..., J$). In the Walrasian general equilibrium theory, $\bar{N}_j = 1$. We measure prices in terms of capital goods. Capital goods is selected to serve as numeriare. We denote the wage rate of worker of type j and rate of interest by $w_j(t)$ and $r(t)$, respectively. Let $p_a(t)$ and $p_s(t)$ denote respectively the prices of agricultural commodity and consumer goods at time t. Land rent $R(t)$ is identical in the economy. Let land be denoted by $L(t)$. Total capital stock $K(t)$ is allocated between the three sectors. We use subscript index, a, i, and s to represent agricultural sector, capital goods sector, and consumers good sectors, respectively. Let $N_m(t)$ and $K_m(t)$ stand for the labor force and capital stocks employed by sector m. Let $L_a(t)$ stand for the land used by the agricultural sector. We use $F_m(t)$ to represent output level of sector m, $m = a, i, s$. The total population $\bar{N}(t)$ and total qualified labor supply $N(t)$ are

$$\bar{N}(t) = \sum_{j=1}^{J} \bar{N}_j(t), \quad N(t) = \sum_{j=1}^{J} h_j(t)\bar{N}_j(t), \tag{1}$$

in which $h_j(t)$ is the human capital of group j. Full employment of labor force is

$$N_a(t) + N_i(t) + N_s(t) = N(t). \tag{2}$$

We introduce

$$k_m(t) \equiv \frac{K_m(t)}{N_m(t)}, \ \ k(t) \equiv \frac{K(t)}{N(t)}, \ \ m = a, i, s.$$

The agricultural sector

There are three input factors, land, labor force, and capital, in the agricultural production. The agricultural sector's production function takes on the following form

$$F_a(t) = A_a(t) K_a^{\alpha_a(t)}(t) N_a^{\beta_a(t)}(t) L_a^{\varsigma(t)}(t), \\ \alpha_a(t), \beta_a(t), \varsigma(t) > 0, \ \ \alpha_a(t) + \beta_a(t) + \varsigma(t) = 1, \tag{3}$$

where $A_a(t)$, $\alpha_a(t)$, $\beta_a(t)$, and $\zeta_a(t)$ are parameters. The marginal conditions imply

$$r(t) + \delta_k(t) = \frac{\alpha_a(t) p_a(t) F_a(t)}{K_a(t)}, \ \ w(t) = \frac{\beta_a(t) p_a(t) F_a(t)}{N_a(t)}, \ \ R(t) = \frac{\varsigma(t) p_a(t) F_a(t)}{L_a(t)}, \tag{4}$$

The capital goods sector

The capital goods sector's production function is specified as follows

$$F_i(t) = A_i(t) K_i^{\alpha_i(t)}(t) N_i^{\beta_i(t)}(t), \ \ \alpha_i(t), \beta_i(t) > 0, \ \ \alpha_i(t) + \beta_i(t) = 1, \tag{5}$$

where $A_i(t)$, $\alpha_i(t)$, and $\beta_i(t)$ are parameters. The marginal conditions for the capital goods sector are given by

$$r(t) + \delta_k(t) = \frac{\alpha_i(t) F_i(t)}{K_i(t)}, \ \ w(t) = \frac{\beta_i(t) F_i(t)}{N_i(t)}. \tag{6}$$

The consumer goods sector

The consumer goods sector's production function takes on the following form

$$F_s(t) = A_s(t) K_s^{\alpha_s(t)}(t) N_s^{\beta_s(t)}(t), \ \ \alpha_s(t) + \beta_s(t) = 1, \ \ \alpha_s(t), \beta_s(t) > 0, \tag{7}$$

where $A_s(t)$, $\alpha_s(t)$, and $\beta_s(t)$ are the technological parameter of the service sector. The marginal conditions are

$$r(t) + \delta_k(t) = \frac{\alpha_s(t) p_s(t) F_s(t)}{K_s(t)}, \ \ w(t) = \frac{\beta_s(t) p_s(t) F_s(t)}{N_s(t)}. \tag{8}$$

Consumer behaviors and wealth dynamics

We apply the model Zhang (2005a). This study applies the public land ownership. The revenue of land is distributed between the population equally. The total land revenue is given by $R(t)L(t)$. We use $\bar{k}_j(t)$ and $\bar{r}(t)$ to represent per household wealth of group j and land revenue. We have

$$\bar{k}_j(t) = \frac{\bar{K}_j(t)}{\bar{N}_j(t)}, \quad \bar{r}(t) = \frac{R(t)L(t)}{\bar{N}(t)},$$

where $\bar{K}_j(t)$ is the total wealth held by group j. Per household's current income from the wage payment $h_j(t)w(t)$, the interest payment $r(t)\bar{k}_j(t)$, and land ownership $\bar{r}(t)$ is

$$y_j(t) = r(t)\bar{k}_j(t) + h_j(t)w(t) + \bar{r}(t).$$

Per household's disposable income is the sum of the current disposable income and the value of wealth

$$\hat{y}_j(t) = y_j(t) + \bar{k}_j(t). \tag{9}$$

A household uses up the disposable income for savings $s_j(t)$, housing (measured by lot size $l_j(t)$), consumption of agricultural goods $c_{aj}(t)$, and consumption of consumer goods $c_{sj}(t)$. The household is subject to the following budget constraint

$$R(t)l_j(t) + p_a(t)c_{aj}(t) + p_s(t)c_{sj}(t) + s_j(t) = \hat{y}_j(t). \tag{10}$$

Utility level $U_j(t)$ is a function of $l_j(t)$, $c_{aj}(t)$, $c_{sj}(t)$, and $s_j(t)$, as follows

$$U_j(t) = l_j^{\eta_{0j}(t)}(t)c_{aj}^{\mu_{0j}(t)}(t)c_{sj}^{\xi_{0j}(t)}(t)s_j^{\lambda_{0j}(t)}(t), \quad \eta_{0j}(t), \mu_{0j}(t), \xi_{0j}(t), \lambda_{0j}(t) > 0,$$

where $\eta_{0j}(t)$ is propensity to consume housing, $\xi_{0j}(t)$ to consume agricultural goods, $\xi_{0j}(t)$ to consume consumer goods, and $\lambda_{0j}(t)$ to save, respectively. The marginal conditions for maximizing the utility function subject to the budget constraints are

$$\begin{aligned} & R(t)l_j(t) = \eta_j(t)\hat{y}_j(t), \quad p_a(t)c_{aj}(t) = \mu_j(t)\hat{y}_j(t), \quad p_s(t)c_{sj}(t) = \xi_j(t)\hat{y}_j(t), \\ & s_j(t) = \lambda_j(t)\hat{y}_j(t), \end{aligned} \tag{11}$$

in which

$$\begin{aligned} & \eta_j(t) \equiv \rho_j(t)\eta_{0j}(t), \quad \mu_j(t) \equiv \rho_j(t)\mu_{0j}(t), \quad \xi_j(t) \equiv \rho_j(t)\xi_{0j}(t), \quad \lambda_j(t) \equiv \rho_j(t)\lambda_{0j}(t), \\ & \rho_j(t) \equiv \frac{1}{\eta_{0j}(t) + \mu_{0j}(t) + \xi_{0j}(t) + \lambda_{0j}(t)}. \end{aligned}$$

The household's change of wealth is savings minus dissavings

$$\dot{\bar{k}}_j(t) = s_j(t) - \bar{k}_j(t) = \lambda_j(t)\hat{y}_j(t) - \bar{k}_j(t). \tag{12}$$

Demand and supply of the three sectors
Equilibrium in agricultural good markets is

$$\sum_{j=1}^{J} c_{aj}(t)\bar{N}_j(t) = F_a(t). \tag{13}$$

Equilibrium in consumer goods markets is

$$\sum_{j=1}^{J} c_{sj}(t)\bar{N}_j(t) = F_s(t). \tag{14}$$

The output of the capital goods sector is used up for the net savings and the depreciation of capital stock

$$S(t) - K(t) + \delta_k(t)K(t) = F_i(t), \tag{15}$$

where

$$S(t) \equiv \sum_{j=1}^{J} s_j(t)\bar{N}_j(t), \quad K(t) = \sum_{j=1}^{J} \bar{k}_j(t)\bar{N}_j(t).$$

Full employment of input factors
The three sectors use up total capital stock $K(t)$

$$K_a(t) + K_i(t) + K_s(t) = K(t). \tag{16}$$

The land is fully used

$$L_a(t) + \sum_{j=1}^{J} l_j(t)\bar{N}_j(t) = L(t). \tag{17}$$

The model is completed. It is a generalization of Zhang's model (Zhang 2017). It is structurally similar to some well-known models (Solow 1956; Uzawa 1961; Samuelson 1959; Pasinetti 1960; Todaro 1969).

3. The dynamics and its properties

The previous developed the model. We now show a computational program to follow movement of the system. The following lemma shows that the dimension of the dynamical system is equal to the number of types of households. We also provide a computational procedure for calculating all the variables at any point in time. First, we define $z(t)$ and $\{\bar{k}_j(t)\}$ by

$$z(t) \equiv \frac{r(t) + \delta_k(t)}{w_1(t)/h_1(t)}, \quad \{\bar{k}_j(t)\} \equiv (\bar{k}_2(t), \cdots, \bar{k}_J(t)),$$

Lemma

The motion of the economic system is given by J differential equations

$$\begin{aligned} \dot{z}(t) &= \Lambda_1(z(t), \{\bar{k}_j(t)\}, t), \\ \dot{\bar{k}}_j(t) &= \Lambda_j(z(t), \{\bar{k}_j(t)\}, t), \quad j = 2, \dots, J, \end{aligned} \tag{18}$$

in which $\Lambda_j(t)$ are functions of $z(t)$, $\{\bar{k}_j(t)\}$ and t defined in the Appendix. Other variables are given as functions of $z(t)$, $\{\bar{k}_j(t)\}$ and t as follows: $r(t)$ and $w_j(t)$ by (A3) $\rightarrow$ $\bar{k}_1(t)$ by (A17) $\rightarrow$ $N_i(t)$ by (A14) $\rightarrow$ $N_a(t)$, $N_s(t)$, and $\bar{r}(t)$ by (A13) $\rightarrow$ $R(t) = \bar{r}(t)\bar{N}(t)/L(t)$ $\rightarrow$ $w(t) = w_1(t)/h_1(t)$ $\rightarrow$ $\hat{y}_j(t)$ by (A6) $\rightarrow$ $K_a(t)$, $K_s(t)$, and $K_i(t)$ by (A1) $\rightarrow$ $F_i(t)$, $F_s(t)$ and $F_a(t)$ by the definitions $\rightarrow$ $p_s(t)$ by (A4) $\rightarrow$ $p_a(t)$ by (4) $\rightarrow$ $l_j(t)$, $c_{sj}(t)$, $c_{aj}(t)$, and $s_j(t)$ by (11) $\rightarrow$ $K(t)$ by (16).

The lemma gives a computational procedure for plotting the motion of the economy with any number of types of households. The rest of this section studies properties of the dynamic system when all the coefficients are constant. The following case is examined by Zhang (2017). We summarize the results. We specify parameter values as follows:

$$A_i = 1.3, \; A_s = 1, \; A_a = 0.8, \; L = 10, \; \alpha_i = 0.34, \; \alpha_s = 0.3, \; \alpha_a = 0.17, \; \beta_a = 0.2,$$
$$\delta_k = 0.05, \; N_1 = 12, \; N_2 = 40, \; N_1 = 20,$$
$$\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1.5 \\ 0.6 \end{pmatrix}, \; \begin{pmatrix} \lambda_{10} \\ \lambda_{20} \\ \lambda_{30} \end{pmatrix} = \begin{pmatrix} 0.78 \\ 0.75 \\ 0.7 \end{pmatrix}, \; \begin{pmatrix} \xi_{10} \\ \xi_{20} \\ \xi_{30} \end{pmatrix} = \begin{pmatrix} 0.12 \\ 0.16 \\ 0.18 \end{pmatrix}, \; \begin{pmatrix} \eta_{10} \\ \eta_{20} \\ \eta_{30} \end{pmatrix} = \begin{pmatrix} 0.2 \\ 0.16 \\ 0.14 \end{pmatrix}, \; \begin{pmatrix} \mu_{10} \\ \mu_{20} \\ \mu_{30} \end{pmatrix} = \begin{pmatrix} 0.06 \\ 0.08 \\ 0.09 \end{pmatrix}.$$

(19)

Initial conditions are as follows

$$z(0) = 0.045, \; \bar{k}_2(0) = 15, \; \bar{k}_3(0) = 8.5.$$

The motion of the variables is plotted in Figure 1. In Figure 1, the national income is

$$Y(t) = F_i(t) + p_s(t)F_s(t) + p_a(t)F_a(t) + (L_0 - L_a(t))R(t).$$

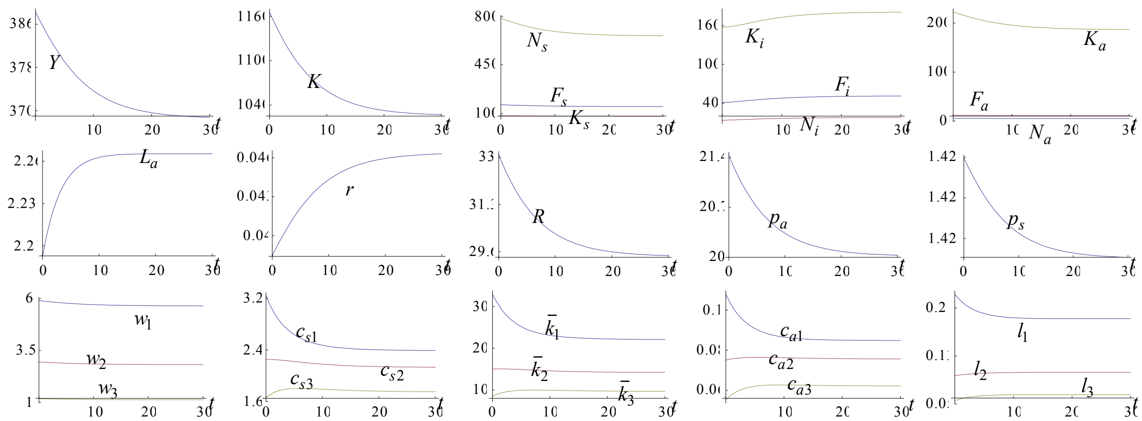

Figure 1. The Motion of the Economic System

We have the equilibrium values of the variables as follows

$$Y = 596.2,\ K = 1025.6,\ F_i = 51.28,\ F_s = 148.9,\ F_a = 5.28,\ r = 0.046,\ R = 29.42,$$
$$p_a = 20.02,\ p_s = 1.42,\ w_1 = 5.64,\ w_2 = 2.82,\ w_3 = 1.13,\ L_a = 2.26,\ N_a = 11.25,$$
$$N_i = 18.0,\ N_s = 78.75,\ K_a = 186.5,\ K_i = 180.9,\ K_s = 658.2,\ \bar{k}_1 = 22.07,\ \bar{k}_2 = 14.18,$$
$$\bar{k}_3 = 9.67,\ c_{s1} = 2.39,\ c_{s2} = 2.13,\ c_{s3} = 1.75,\ c_{a1} = 0.085,\ c_{a2} = 0.075,\ c_{a3} = 0.062,$$
$$l_1 = 0.192,\ l_2 = 0.103,\ l_3 = 0.066. \qquad (20)$$

The three eigenvalues are

$$\{-0.335,\ -0.303,\ -0.159\}.$$

The equilibrium point is locally stable. This guarantees validity of comparative dynamic analysis in the next section.

4. Comparative dynamic analysis

We showed the movement of the national economy when coefficients are constant. This section shows oscillations around the paths plotted Figure 1. We apply variable $\overline{\Delta} x_j(t)$ to represent the change rate of variable, $x_j(t)$, in percentage due to changes in parameter values.

4.1. Group 1' propensity to save periodically oscillates

It is essential to take account in variety of preferences in order to properly model economic structural change and economic development. Nevertheless, Walrasian general equilibrium theory has failed to provide a proper analytical framework for examining important issues related to growth and distribution. Our Walrasian equilibrium model with endogenous capital enables to examine how any exogenous time-dependent shocks affect growth and income and wealth distribution. We now examine a case that group 1' propensity to save periodically oscillates as follows:

$\lambda_{01}(t) = 0.78 + 0.01\sin(t).$

Figure 2 plots the simulation results. The national economy experiences business cycles. The variables related to group 1's oscillate more greatly than the corresponding variables of the other two groups.

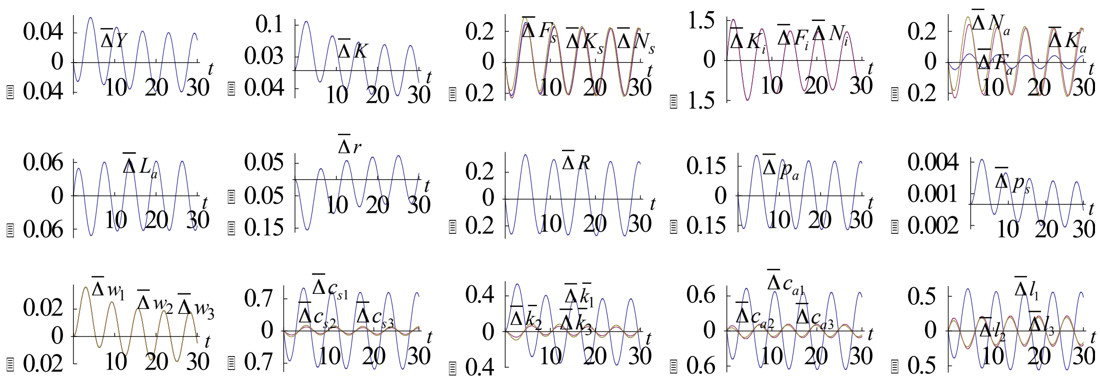

Figure 2. Group 1' Propensity to Save Periodically Oscillates

4.2. Group 1's human capital periodically oscillates

Human capital is an important determinant of economic growth. We now examine whether exogenous changes in human capital may result in business cycles. We now allow group 1's human capital to periodically oscillate as follows:

$$h_1(t) = 3 + 0.1\sin(t).$$

We plot the simulation results in Figure 3. Time-dependent perturbations in group 1's human capital results in business cycles in the national economy.

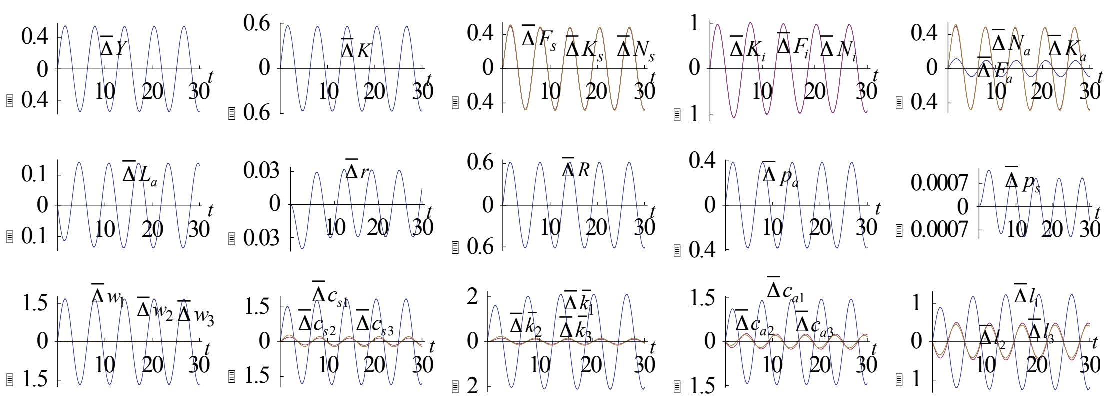

Figure 3. Group 1's Human Capital Periodically Oscillates

4.3. Group 3's propensity to consume agricultural good periodically oscillates

We now examine how group 3's preference for agricultural good may affect the economic structure and growth. We allow group 3 to increase its propensity to consume agricultural good as follows: $\mu_{03} : 0.09 \Rightarrow 0.11$.

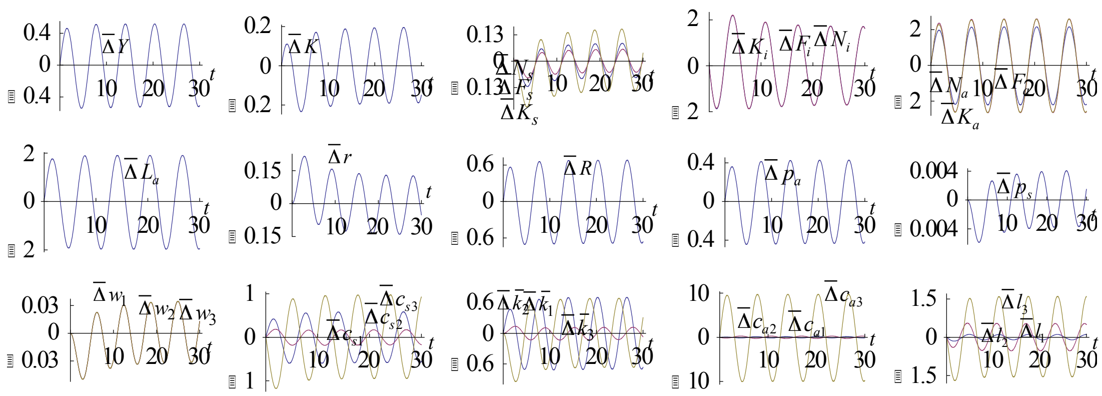

Figure 4. Group 3's Propensity to Consume Agricultural Good Periodically Oscillates

4.4. Group 3's population periodically oscillates

There are many theoretical growth models with human capital and population (e.g., Galor and Weil 1999; Bretschger 2013). There are mixed conclusions from empirical studies related to issues (e.g., Furuoka 2009; Yao *et al*. 2013). Although this study does not analyze endogenous change in population, we show that periodic population change results in business cycles. We consider a case that group 3'*s* population periodically oscillates as follows:

$$N_3(t) = 20 + \sin(t).$$

Figure 5 plots the simulation results

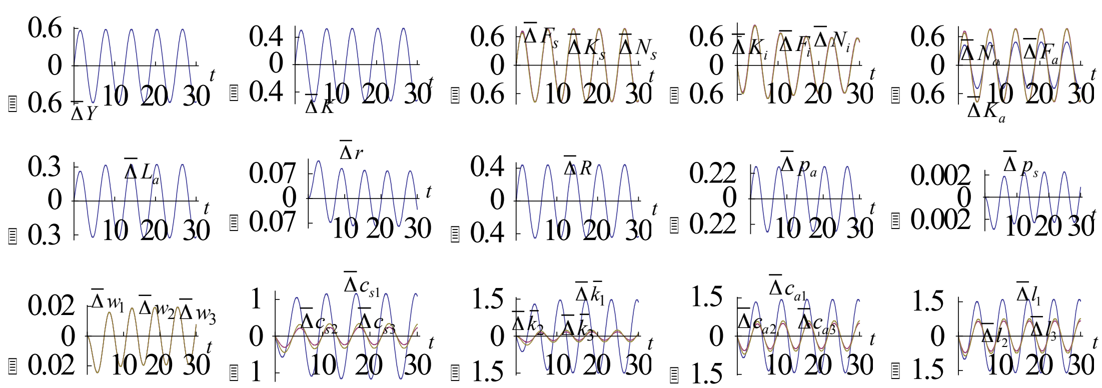

Figure 5. Group 3's Population Periodically Oscillates

5. Concluding Remarks

This study generalized the economic growth model of heterogeneous households proposed by Zhang (2017). Zhang's model is built on three well-known Ricardian theory of distribution, Walrasian general equilibrium theory, and neoclassical growth theory. The heterogeneous-household economy is composed of one consumer goods sector, one agricultural goods sector, and one capital goods sector. Technology, land, population and human capital are exogenous. This paper generalizes Zhang's model by allowing constant coefficients to be time-dependent. We showed existence of business cycles due to different exogenous oscillatory changes. As it is developed on the basis of some well-known theoretical models and each of these models has a vast literature of its extensions and generalizations, our model may be generalized and extended in different ways.

Appendix: Proving the lemma

From (4), (6), and (8), we get

$$z \equiv \frac{r+\delta_k}{w} = \frac{N_a}{\bar{\beta}_a K_a} = \frac{N_i}{\bar{\beta}_i K_i} = \frac{N_s}{\bar{\beta}_s K_s}, \tag{A1}$$

in which $\bar{\beta}_j \equiv \beta_m / \alpha_m$. Equations (A1) and (15) imply

$$\bar{\beta}_a K_a + \bar{\beta}_i K_i + \bar{\beta}_s K_s = \frac{N}{z}. \tag{A2}$$

Substitute (A1) into (4)

$$r = \alpha_r z^{\beta_i} - \delta_k, \;\; w_j = \alpha_j z^{-\alpha_i}, \tag{A3}$$

in which

$$\alpha_r = \alpha_i A_i \bar{\beta}_i^{\beta_i}, \;\; \alpha_j = h_j \beta_i A_i \bar{\beta}_i^{-\alpha_i}.$$

By (7) and (8), we get

$$p_s = \frac{\bar{\beta}_s^{\alpha_s} z^{\alpha_s} w}{\beta_s A_s}. \tag{A4}$$

Equation (4) implies

$$L_a = \frac{w \varsigma N_a}{\beta_a R}. \tag{A5}$$

With (A3) and the definitions of $\hat{y}_j$, we get

$$\hat{y}_j = (1+r)\bar{k}_j + h_j w + \bar{r}. \tag{A6}$$

Substitute $p_s c_{sj} = \xi_j \hat{y}_j$ into (13)

$$\sum_{j=1}^{J} \xi_j \bar{N}_j \hat{y}_j = p_s F_s. \tag{A7}$$

From (A6) and (A7) we have

$$\frac{wN_s}{\beta_s} - \bar{r}\,\tilde{g}_0 = \sum_{j=1}^{J} \tilde{g}_j\,\bar{k}_j + \tilde{g}\,w, \tag{A8}$$

in which we apply $p_s F_s = wN_s / \beta_s$ and

$$\tilde{g}_j(z) \equiv (1 + r)\xi_j\,\bar{N}_j, \;\; \tilde{g} \equiv \sum_{j=1}^{J} h_j\,\xi_j\,\bar{N}_j, \;\; \tilde{g}_0 \equiv \sum_{j=1}^{J} \xi_j\,\bar{N}_j.$$

Substitute $p_a c_{aj} = \mu_j\,\hat{y}_j$ in (11) into (13)

$$\sum_{j=1}^{J} \mu_j\,\hat{y}_j\,\bar{N}_j = \frac{wN_a}{\beta_a}, \tag{A9}$$

in which we apply $p_a F_a = wN_a / \beta_a$. Insert (A6) in (A9)

$$\frac{wN_a}{\beta_a} - \tilde{h}_0\,\bar{r} = \sum_{j=1}^{J} \tilde{h}_j\,\bar{k}_j + w\tilde{h}, \tag{A10}$$

in which

$$\tilde{h}_j(z) \equiv (1 + r)\mu_j\,\bar{N}_j, \;\; \tilde{h} \equiv \sum_{j=1}^{J} h_j\,\mu_j\,\bar{N}_j, \;\; \tilde{h}_0 \equiv \sum_{j=1}^{J} \mu_j\,\bar{N}_j.$$

With (A5), $Rl_j = \eta_j\,\hat{y}_j$ from (11), and (17), we get

$$\frac{w\varsigma N_a}{\beta_a} + \sum_{j=1}^{J} \eta_j\,\hat{y}_j\,\bar{N}_j = RL. \tag{A11}$$

By (A6) in (A11), we have

$$-\frac{w\varsigma N_a}{\beta_a} + \left(\bar{N} - \bar{h}_0\right)\bar{r} = \sum_{j=1}^{J} \bar{h}_j\,\bar{k}_j + \bar{h}\,w, \tag{A12}$$

in which

$$\bar{h}_j(z) \equiv (1 + r)\eta_j\,\bar{N}_j, \;\; \bar{h} \equiv \sum_{j=1}^{J} h_j\,\eta_j\,\bar{N}_j, \;\; \bar{h}_0 \equiv \sum_{j=1}^{J} \eta_j\,\bar{N}_j.$$

It is straightforward to solve (A8), (A10) and (A12) with N_a, N_s, and $\bar{r}$ as variables as follows

$$N_a\left(z, \left(\bar{k}_j\right), t\right) = \sum_{j=1}^{J} \varphi_{aj}(z, t)\bar{k}_j,$$
$$N_s\left(z, \left(\bar{k}_j\right), t\right) = \sum_{j=1}^{J} \varphi_{sj}(z, t)\bar{k}_j, \quad \text{(A13)}$$
$$\bar{r}\left(z, \left(\bar{k}_j\right), t\right) = \sum_{j=1}^{J} \varphi_{Rj}(z, t)\bar{k}_j.$$

Explicit expressions of $\varphi_{..}(z, t)$ are not given as they are tedious. Equations (2) and (A13) imply

$$N_i\left(z, \left(\bar{k}_j\right), t\right) = \sum_{j=1}^{J} \varphi_{ij}(z,t)\bar{k}_j. \quad \text{(A14)}$$

By (16) , (A1), and $K = \sum_{j=1}^{J} \bar{k}_j \, \bar{N}_j$, we obtain

$$\frac{N_a}{\bar{\beta}_a} + \frac{N_i}{\bar{\beta}_i} + \frac{N_s}{\bar{\beta}_s} = z \sum_{j=1}^{J} \bar{k}_j \, \bar{N}_j. \quad \text{(A15)}$$

Insert N_a, N_s, and N_i in (A13) in (A14) yields

$$\sum_{j=1}^{J} \varphi_j(z, t)\bar{k}_j = 0, \quad \text{(A16)}$$

in which

$$\varphi_j(z, t) \equiv \frac{\varphi_{aj}}{\bar{\beta}_a} + \frac{\varphi_{ij}}{\bar{\beta}_i} + \frac{\varphi_{sj}}{\bar{\beta}_s} - z\,\bar{N}_j.$$

Solve (A16) in $\bar{k}_1$

$$\bar{k}_1 = \phi\left(z, \left\{\bar{k}_j\right\}, t\right) \equiv -\frac{1}{\phi_1} \sum_{j=2}^{J} \phi_j \, \bar{k}_j. \quad \text{(A17)}$$

We now confirmed the computational procedure in the Lemma. With the procedure, (A21), and (12), we get

$$\dot{\bar{k}}_1 = \bar{\Omega}_1\left(z, \left\{\bar{k}_j\right\}, t\right) \equiv \lambda_1 \, \hat{y}_1 - \varphi, \quad \text{(A18)}$$

$$\dot{\bar{k}}_j = \Lambda_j\left(z, \{\bar{k}_j\}, t\right) \equiv \lambda_j \bar{y}_j - \bar{k}_j, \quad j = 3, \ldots, J. \tag{A19}$$

Taking derivatives of equation (A17) in t and applying (A18), we have

$$\dot{\bar{k}}_1 = \frac{\partial \varphi}{\partial z}\dot{z} + \frac{\partial \varphi}{\partial t} + \sum_{j=2}^{J} \Lambda_j \frac{\partial \varphi}{\partial \bar{k}_j}. \tag{A20}$$

Equal the right-hand sides of equations (A18) and (A20)

$$\dot{z} = \left[\overline{\Omega}_1 - \frac{\partial \varphi}{\partial t} - \sum_{j=2}^{J} \Lambda_j \frac{\partial \varphi}{\partial \bar{k}_j}\right]\left(\frac{\partial \varphi}{\partial z}\right)^{-1}. \tag{A21}$$

In summary, we confirmed the Lemma.

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